Logarithmic Voronoi cells

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SFSU Algebra, Geometry, and Combinatorics Seminar
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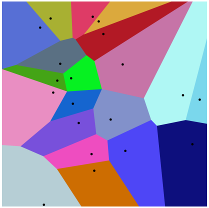
Voronoi cells in the Euclidean case

from Wikipedia:



Let X be a **finite** point configuration in $\mathbb{R}^n$.

- The *Voronoi cell* of $x \in X$ is the set of all points that are closer to x than any other $y \in X$, in the Euclidean metric.
- The subset of points that are equidistant from x and any other points in X is the *boundary* of the Voronoi cell of x.
- Voronoi cells partition $\mathbb{R}^n$ into convex polyhedra.

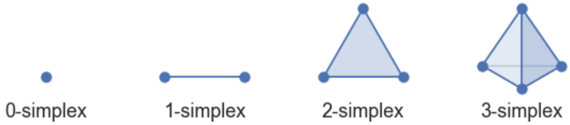
If X is a **variety**, each Voronoi cell is a convex semialgebraic set in the normal space of X at a point. The algebraic boundaries of these Voronoi cells were computed by Cifuentes, Ranestad, Sturmfels and Weinstein.

Log-Voronoi cells for discrete models

We explore Voronoi cells in the context of algebraic statistics.

- A *probability simplex* is defined as

$$\Delta_{n-1} = \{(p_1, \ldots, p_n) \,:\, p_1 + \cdots + p_n = 1, p_i \geq 0 \text{ for } i \in [n]\}.$$



- A *statistical model* $\mathcal{M}$ is a subset of a probability simplex.
- An *algebraic statistical model* is a subset $\mathcal{M} = \mathcal{V}(f) \cap \Delta_{n-1}$ for some polynomial system of equations $f : \mathbb{C}^n \to \mathbb{C}^m$.
- For an empirical data point $u = (u_1, ..., u_n) \in \Delta_{n-1}$, the *log-likelihood function* defined by u assuming distribution $p = (p_1, ..., p_n) \in \mathcal{M}$ is

$$\ell_u(p) = u_1 \log p_1 + u_2 \log p_2 + \cdots + u_n \log p_n + \log(c).$$

Ice Cream! 

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(p_1, p_2, p_3)

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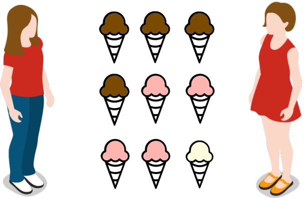

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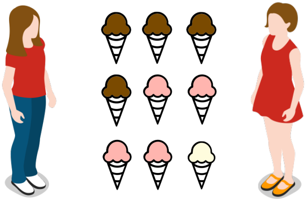

$$L = c \cdot p_1^{4/9} p_2^{4/9} p_3^{1/9}$$

$$\ell_u = 4/9 \cdot \log(p_1) + 4/9 \cdot \log(p_2) + 1/9 \cdot \log(p_3) + \log(c).$$

Log-Voronoi cells

There are two natural problems to consider:

1. The maximum likelihood estimation problem (MLE):

Given a sampled empirical distribution $u \in \Delta_{n-1}$, which point $p \in \mathcal{M}$ did it most likely come from? In other words, we wish to maximize $\ell_u(p)$ over all points $p \in \mathcal{M}$.

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2. Computing logarithmic Voronoi cells:

Given a point in the model $q \in \mathcal{M}$, what is the set of all points $u \in \Delta_{n-1}$ that have q as a global maximum when optimizing the function ℓ_u?

We call the set of all such elements $u \in \Delta_{n-1}$ above the *logarithmic Voronoi cell* of q.

Log-normal spaces and polytopes

Suppose our algebraic statistical model $\mathcal{M}$ is given by the vanishing set of the polynomial system $f = (f_1, \cdots, f_m)$. Let $u \in \Delta_{n-1}$ be fixed.

- The method of *Lagrange multipliers* can be used to find critical points of $\ell_u(x) = u_1 \log x_1 + u_2 \log x_2 + \cdots + u_n \log x_n$ given the constraints f.
- We form the *augmented Jacobian*:

$$A = \begin{bmatrix} \mathcal{J}_f \\ \nabla \ell_u \end{bmatrix} = \begin{bmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \\ \nabla \ell_u \end{bmatrix}$$

- All $(c+1) \times (c+1)$ minors of A must vanish, where c is the co-dimension of $\mathcal{M}$.

Log-normal spaces and polytopes

Fix some point $q \in \mathcal{M}$ and let u vary.

- Vanishing of $(c+1) \times (c+1)$ minors is a linear condition in u_i.
- The *log-normal space* of q is the *linear* space of possible data points u that have a chance of getting mapped to q via the MLE (all points at which all minors vanish).

$$\log \mathcal{N}_q(\mathcal{M}) = \{u_1 \boldsymbol{v}_1 + \cdots + u_n \boldsymbol{v}_n : u \in \mathbb{R}^n\} \text{ for some fixed } \boldsymbol{v}_i \in \mathbb{R}^n.$$

- Intersecting $\log \mathcal{N}_q$ with the simplex Δ_{n-1}, we obtain a polytope, which we call the *log-normal polytope* of q.
- The log-normal polytope of q contains the log-Voronoi cell of q.

The Hardy-Weinberg curve

Consider a model parametrized by

$$p \mapsto \left(p^2, 2p(1-p), (1-p)^2\right).$$

Performing implicitization, we find that the model $\mathcal{M} = \mathcal{V}(f)$ where $f : \mathbb{C}^3 \to \mathbb{C}^2$ is given by:

$$f = \begin{bmatrix} 4x_1x_3 - x_2^2 \\ x_1 + x_2 + x_3 - 1 \end{bmatrix}.$$

The augmented Jacobian is given by:

$$A = \begin{bmatrix} 4x_3 & -2x_2 & 4x_1 \\ 1 & 1 & 1 \\ u_1/x_1 & u_2/x_2 & u_3/x_3 \end{bmatrix}.$$

Fix a point $q \in \mathcal{M}$ and substitute x_i for q_i in A. All points $u \in \mathbb{R}^3$ at which the determinant vanishes define the log-normal space at q.

The Hardy-Weinberg curve

$$\det A = 4u_1 - 4u_3 - 4u_2 \cdot \frac{x_1}{x_2} + 2u_1 \cdot \frac{x_2}{x_1} - 2u_3 \cdot \frac{x_2}{x_3} + 4u_2 \cdot \frac{x_3}{x_2}$$

For example, at $p = 0.2$, we get a point $q = (0.04, 0.32, 0.64) \in \mathcal{M}$. The log-normal space at q is the plane

$$20u_1 + 7.5u_2 - 5u_3 = 0.$$

Sampling more points, we get the following pictures:

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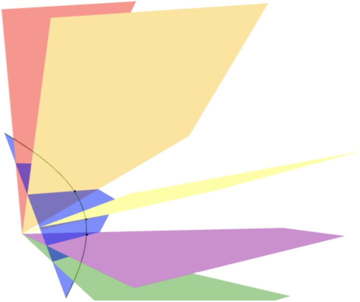

Log-normal spaces

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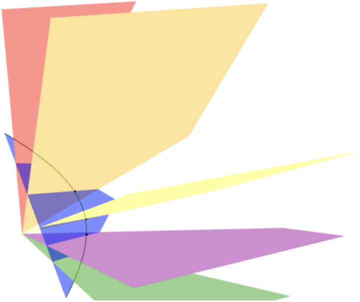

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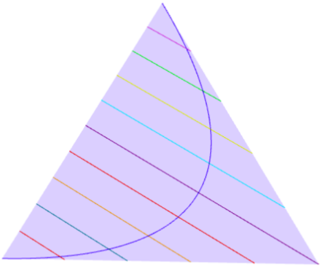

Log-normal polytopes = Log-Voronoi cells

Two-bits independence model

Consider a model parametrized by

$$(p_1, p_2) \mapsto \begin{bmatrix} p_1 p_2 \\ p_1(1-p_2) \\ (1-p_1)p_2 \\ (1-p_1)(1-p_2) \end{bmatrix}.$$

Computing the elimination ideal, we get $\mathcal{M} = \mathcal{V}(f)$ where

$$f = \begin{bmatrix} x_1x_4 - x_2x_3 \\ x_1 + x_2 + x_3 + x_4 - 1 \end{bmatrix}.$$

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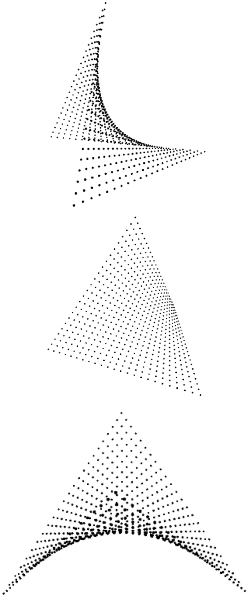

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For any point $q = (q_1, q_2, q_3, q_4) \in \mathcal{M}$. The four 3×3 minors at q are given by

$$\begin{array}{c} u_2 - u_3 - \frac{u_1 q_2}{q_1} + \frac{u_1 q_3}{q_1} + \frac{u_2 q_4}{q_2} - \frac{u_3 q_4}{q_3} \\ u_1 - u_4 - \frac{u_2 q_1}{q_2} + \frac{u_1 q_3}{q_1} - \frac{u_4 q_3}{q_4} + \frac{u_2 q_4}{q_2} \\ u_1 - u_4 + \frac{u_1 q_2}{q_1} - \frac{u_3 q_1}{q_3} - \frac{u_4 q_2}{q_4} + \frac{u_3 q_4}{q_3} \\ u_2 - u_3 + \frac{u_2 q_1}{q_2} - \frac{u_3 q_1}{q_3} - \frac{u_4 q_2}{q_4} + \frac{u_4 q_3}{q_4}. \end{array}$$

The log normal space at q is parametrized as

$$u_3 \begin{pmatrix} \frac{q_1^2 - q_1 q_4}{(q_1+q_2)q_3} \\ \frac{q_1 q_2 + q_2 q_3}{(q_1+q_2)q_3} \\ 1 \\ 0 \end{pmatrix} + u_4 \begin{pmatrix} \frac{q_1 q_2 + q_1 q_4}{(q_1+q_2)q_4} \\ \frac{q_2^2 - q_2 q_3}{(q_1+q_2)q_4} \\ 0 \\ 1 \end{pmatrix}.$$

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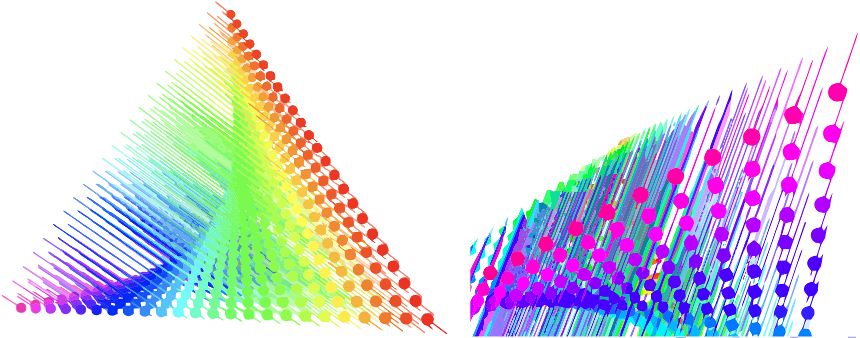

Twisted cubic

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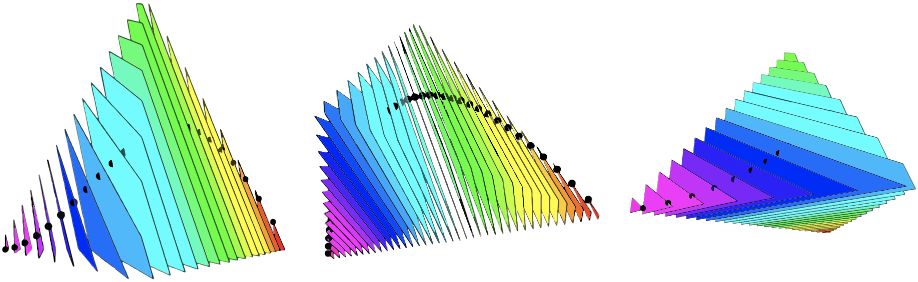

When are log-Voronoi cells polytopes?

If $\mathcal{M}$ is a finite model, then logarithmic Voronoi cells $\log\mathrm{Vor}_{\mathcal{M}}(p)$ are polytopes for each $p \in \mathcal{M}$.

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Let $\Theta \subseteq \mathbb{R}^d$ be a parameter space. Suppose $\mathcal{M}$ is given by

$$f : \Theta \to \Delta_{n-1} : (\theta_1, \cdots, \theta_d) \mapsto (f_1(\theta), \cdots, f_n(\theta)).$$

Then $\ell_u(p) = \sum_{i=1}^n u_i \log f_i(\theta)$. The *likelihood equations* are

$$\sum_{i=1}^n \frac{u_i}{f_i} \cdot \frac{\partial f_i}{\partial \theta_j} = 0 \text{ for } j \in [d].$$

The *maximum likelihood degree* (ML degree) of $\mathcal{M}$ is the number of complex solutions to the likelihood equations for generic data u.

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The *maximum likelihood degree* (ML degree) of $\mathcal{M}$ is the number of complex solutions to the likelihood equations for generic data u.

If $\mathcal{M}$ is a model of *ML degree 1*, then the logarithmic Voronoi cell at every $p \in \mathcal{M}$ equals its log-normal polytope.

When are log-Voronoi cells polytopes?

A discrete *linear model* is given parametrically by nonzero linear polynomials.

Theorem (A., Heaton)

Let $\mathcal{M}$ be a linear model. Then the logarithmic Voronoi cells are equal to their log-normal polytopes.

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Let $\mathcal{M}$ be a linear model. Then the logarithmic Voronoi cells are equal to their log-normal polytopes.

For an $m \times n$ integer matrix A with $\mathbf{1} \in \text{rowspan}(A)$, the corresponding *toric model* $\mathcal{M}_A$ is defined to be the set of all points $p \in \Delta_{n-1}$ such that $\log(p) \in \text{rowspan}(A)$.

Theorem (A., Heaton)

Let A be an integer matrix with $\mathbf{1}$ in its row span and let $\mathcal{M}_A$ be the associated toric model. Then for any point $p \in \mathcal{M}$, the log-Voronoi cell of p is equal to the log-normal polytope at p.

Discrete linear models

Any d-dimensional linear model inside Δ_{n-1} can be written as

$$\mathcal{M} = \{c - Bx : x \in \Theta\}$$

where B is a $n \times d$ matrix, whose columns sum to 0, and $c \in \mathbb{R}^n$ is a vector, whose coordinates sum to 1.

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A *co-circuit* of B is a vector $v \in \mathbb{R}^n$ of minimal support such that $vB = 0$. A co-circuit is *positive* if all its coordinates are positive.

We call a point $p = (p_1, \ldots, p_n) \in \mathcal{M}$ is *interior* if $p_i > 0$ for all $i \in [n]$.

How can we describe logarithmic Voronoi cells of interior points in $\mathcal{M}$?

Interior points

For an interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell at p is the set

$$\log \mathrm{Vor}_{\mathcal{M}}(p) = \left\{ r \cdot \mathrm{diag}(p) \in \mathbb{R}^n : rB = 0,\ r \geq 0,\ \sum_{i=1}^{n} r_i p_i = 1 \right\}.$$

Proposition

For any interior point $p \in \mathcal{M}$, the vertices of $\log \mathrm{Vor}_{\mathcal{M}}(p)$ are of the form $v \cdot \mathrm{diag}(p)$ where v are unique representatives of the positive co-circuits of B such that $\sum_{i=1}^{n} v_i p_i = 1$.

Gale diagrams

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a vector configuration in $\mathbb{R}^d$, whose affine hull has dimension d. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}.$$

Let $\{B_1, \ldots, B_{n-d-1}\}$ be a basis for $\ker(A)$ and $B := [B_1 \; B_2 \; \cdots \; B_{n-d-1}]$. The configuration $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-d-1}\}$ of row vectors of B is the *Gale diagram* of $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$.

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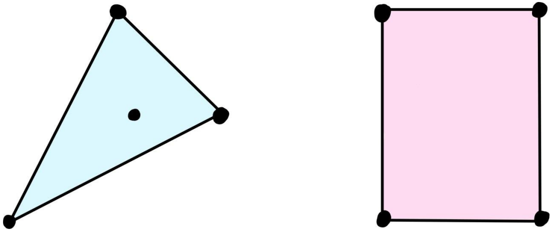
Theorem (A.)

For any interior point $p \in \mathcal{M}$, the logarithmic Voronoi cell of p is combinatorially isomorphic to the dual of the polytope obtained by taking the convex hull of a vector configuration with Gale diagram B.

Corollary

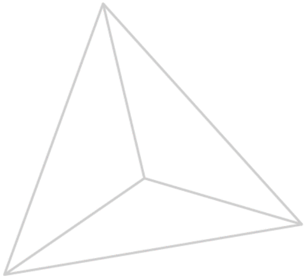
Logarithmic Voronoi cells of all interior points in a linear models have the same combinatorial type.



Proposition

Every $(n-d-1)$-dimensional polytopes with at most n facets appears as a log-Voronoi cell of a d-dimensional linear model inside Δ_{n-1}.

Examples



$B = [1, -5, 3, 1]^T$
$c = (1/4, 1/4, 1/4, 1/4)$

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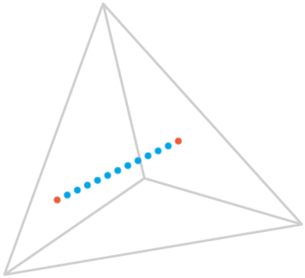

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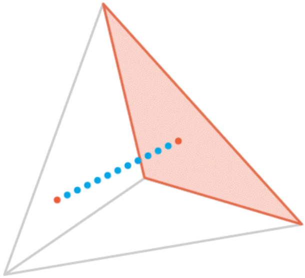

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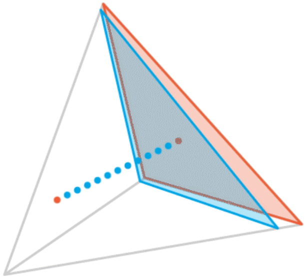

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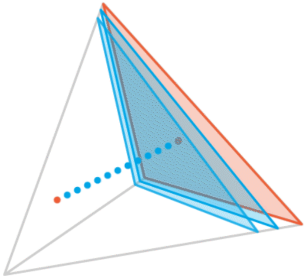

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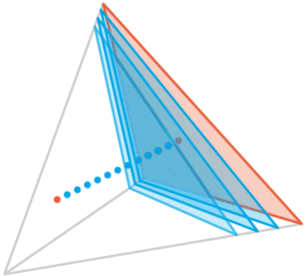

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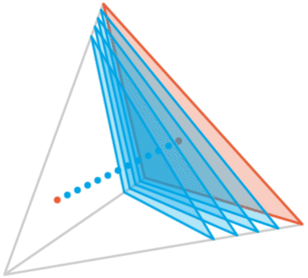

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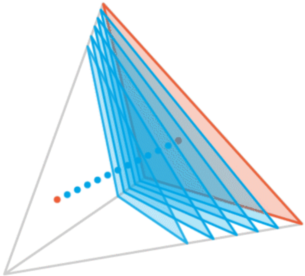

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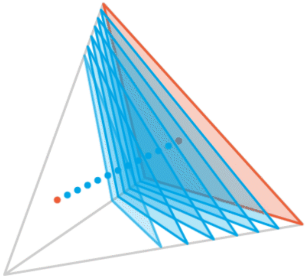

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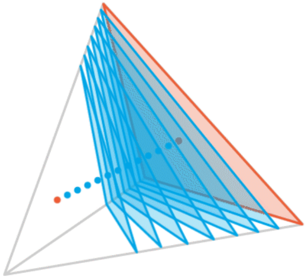

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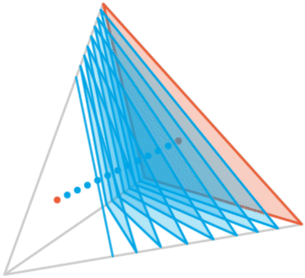

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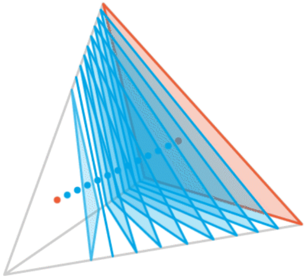

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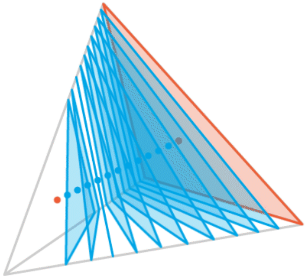

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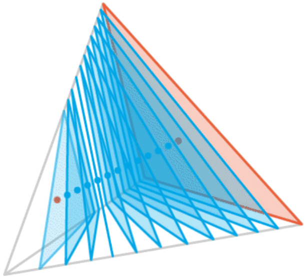

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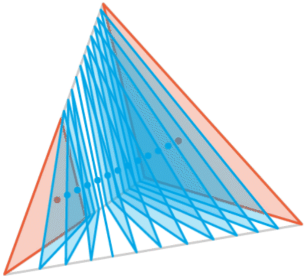

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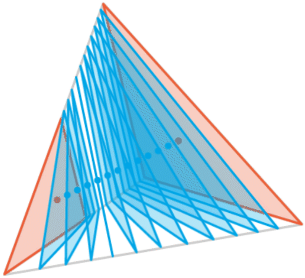

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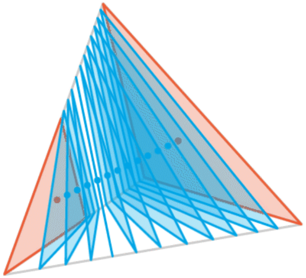

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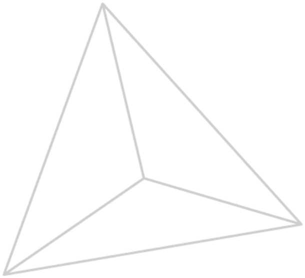

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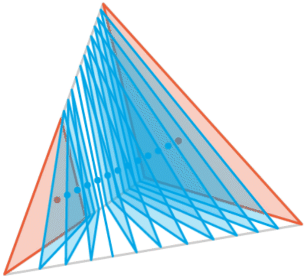

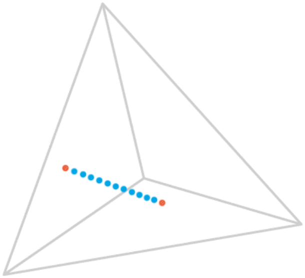

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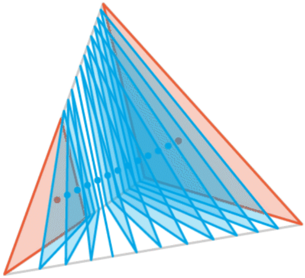

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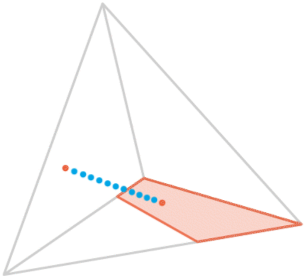

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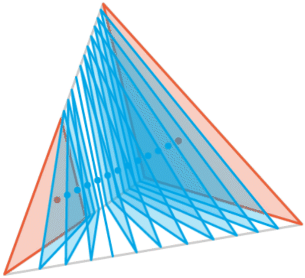

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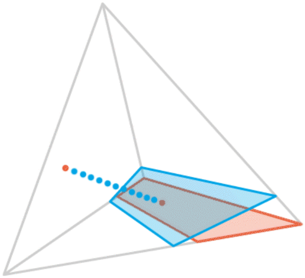

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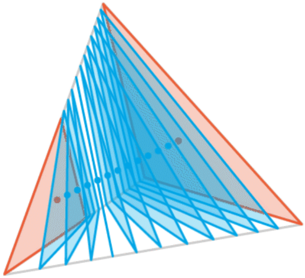

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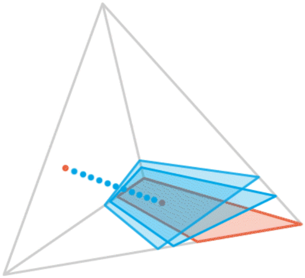

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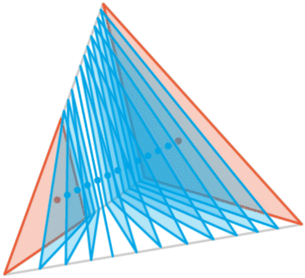

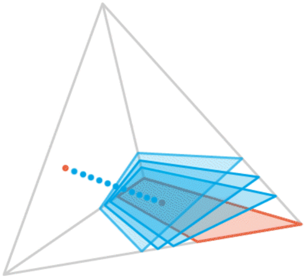

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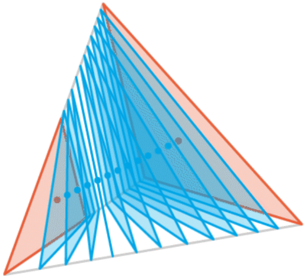

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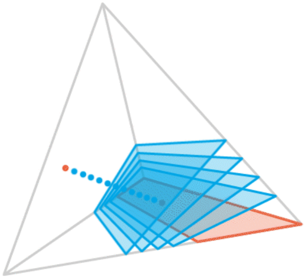

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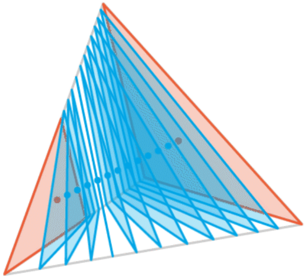

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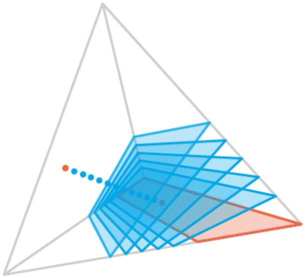

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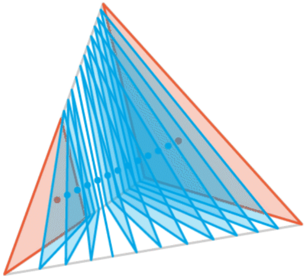

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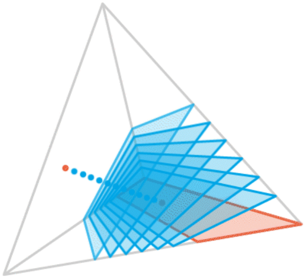

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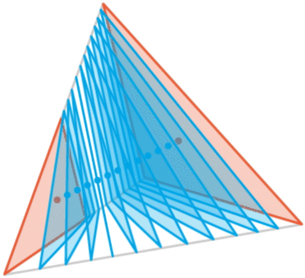

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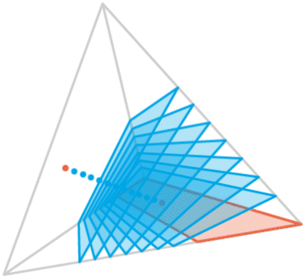

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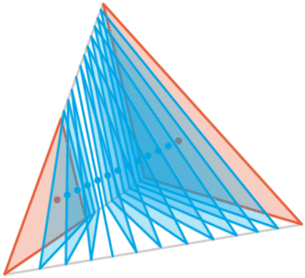

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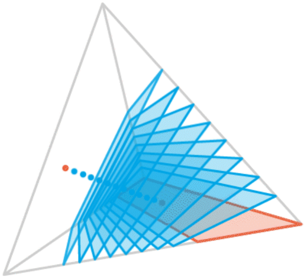

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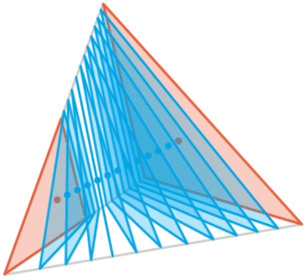

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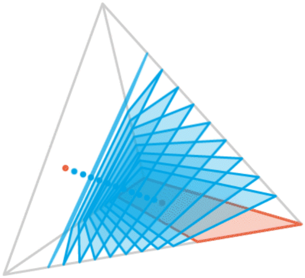

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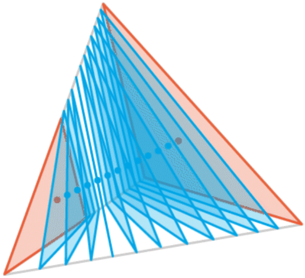

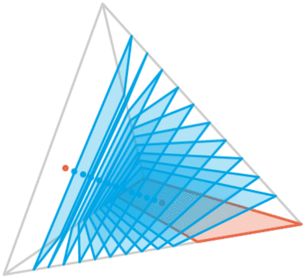

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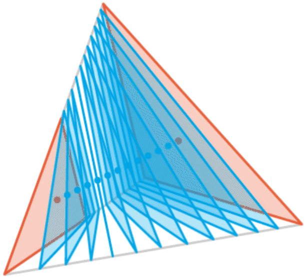

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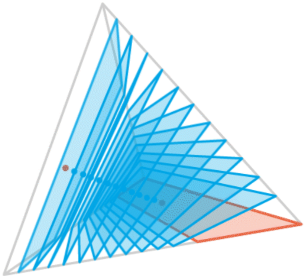

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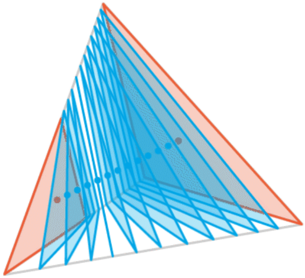

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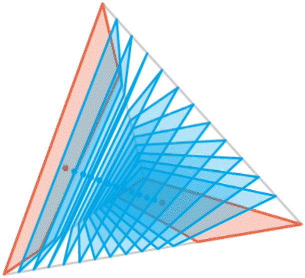

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On the boundary

Let $\mathcal{M}$ be a 1-dimensional linear model inside the simplex Δ_{n-1}. Then $\mathcal{M} = \{c - Bx : x \in \Theta\}$, where

$$B = [\underbrace{b_1 \ \dots \ b_m}_{>0} \ \underbrace{b_{m+1} \ \dots \ b_n}_{<0}]^T \text{ and } c = (c_i).$$

Then Θ is the interval $[x_\ell, x_r] = [c_\ell/b_\ell, c_r/b_r]$ where $b_\ell < 0$ and $b_r > 0$. The log-Voronoi cell at x_r is the polytope at the boundary of Δ_{n-1} with the vertices

$$\{e_j : b_j < 0\} \cup \left\{ \underbrace{\frac{(c_i - b_i(c_r/b_r))b_j}{b_jc_i - b_ic_j} e_i - \frac{(c_j - b_j(c_r/b_r))b_i}{b_jc_i - b_ic_j} e_j}_{v_{ij}} : \substack{i \neq r, \\ b_i > 0, \\ b_j < 0} \right\}.$$

The vertex v_{ij} degenerates into the vertex e_j iff $M_{ri} = 0$, where $M = [B \ c]$. The log-Voronoi cell at x_ℓ is described similarly.

Non-polytopal example

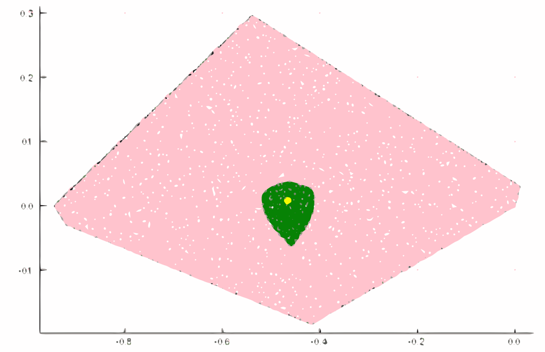
- $\mathcal{M}$ is a 3-dimensional model inside the 5-dimensional simplex given by:

$$
\begin{aligned}
f_0 &= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 - 1 \\
f_1 &= 20x_0x_2x_4 - 10x_0x_3^2 - 8x_1^2x_4 + 4x_1x_2x_3 - x_2^3 \\
f_2 &= 100x_0x_2x_5 - 20x_0x_3x_4 - 40x_1^2x_5 + 4x_1x_2x_4 + 2x_1x_3^2 - x_2^2x_3 \\
f_3 &= 100x_0x_3x_5 - 40x_0x_4^2 - 20x_1x_2x_5 + 4x_1x_3x_4 + 2x_2^2x_4 - x_2x_3^2 \\
f_4 &= 20x_1x_3x_5 - 8x_1x_4^2 - 10x_2^2x_5 + 4x_2x_3x_4 - x_3^3
\end{aligned}
$$

- Pick point $p = \left(\frac{518}{9375}, \frac{124}{625}, \frac{192}{625}, \frac{168}{625}, \frac{86}{625}, \frac{307}{9375}\right) \in \mathcal{M}$.
- 225 4×4 minors of augmented Jacobian define the log-normal space.

Non-polytopal example

- Log-normal space of p is 3-dimensional, and the log-normal polytope of p is a hexagon.
- Using the numerical Julia package HomotopyContinuation.jl, we may compute the logarithmic Voronoi cell of p:



(joint work with Alex Heaton and Sascha Timme)

Continuous statistical models

Let X be an m-dimensional random vector, which has the density function

$$p_{\mu,\Sigma}(x) = \frac{1}{(2\pi)^{m/2}(\det \Sigma)^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}, \quad x \in \mathbb{R}^m$$

with respect to the parameters $\mu \in \mathbb{R}^m$ and $\Sigma \in \mathrm{PD}_m$.

Such X is distributed according to the *multivariate normal distribution*, also called the *Gaussian distribution* $\mathcal{N}(\mu, \Sigma)$.

For $\Theta \subseteq \mathbb{R}^m \times \mathrm{PD}_m$, the statistical model

$$\mathcal{P}_\Theta = \{\mathcal{N}(\mu, \Sigma) : \theta = (\mu, \Sigma) \in \Theta\}$$

is called a *Gaussian model*.

Gaussian models

For a sampled data consisting of n vectors $X^{(1)}, \cdots, X^{(n)} \in \mathbb{R}^m$, we define the *sample mean* and *sample covariance* as

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X^{(i)} \qquad \text{and} \qquad S = \frac{1}{n}\sum_{i=1}^{n}(X^{(i)} - \bar{X})(X^{(i)} - \bar{X})^T,$$

respectively. The *log-likelihood function* is defined as

$$\ell_n(\mu, \Sigma) = -\frac{n}{2}\log\det\Sigma - \frac{1}{2}\operatorname{tr}\left(S\Sigma^{-1}\right) - \frac{n}{2}(\bar{X} - \mu)^T\Sigma^{-1}(\bar{X} - \mu).$$

The problem of maximizing $\ell_n(\Sigma)$ over Θ is *maximum likelihood estimation*.

The *logarithmic Voronoi cell* of $\theta = (\mu, \Sigma) \in \Theta$, is the set of all multivariate distributions $(\bar{X}, S)$ for which ℓ_n is maximized at θ.

Gaussian models

Proposition

Consider the Gaussian model with parameter space $\Theta = \Theta_1 \times \{Id_m\}$ for some $\Theta_1 \subseteq \mathbb{R}^m$. For any point in this model, its logarithmic Voronoi cell is equal to its Euclidean Voronoi cell.

In practice, we consider models given by parameter spaces of the form $\Theta = \mathbb{R}^m \times \Theta_2$ where $\Theta_2 \subseteq \mathrm{PD}_m$. The log-likelihood function is then

$$\ell_n(\Sigma, S) = -\frac{n}{2} \log \det \Sigma - \frac{n}{2} \operatorname{tr}(S\Sigma^{-1}).$$

For $\Sigma \in \Theta_2$, the *log-normal matrix space* $\mathcal{N}_\Sigma \Theta_2$ at Σ is the set of $S \in \mathrm{Sym}_m(\mathbb{R})$ such that Σ appears as a critical point of $\ell_n(\Sigma, S)$. The intersection $\mathrm{PD}_m \cap \mathcal{N}_\Sigma \Theta_2$ is the *log-normal spectrahedron* of Σ.

If Σ is a covariance matrix, its inverse Σ^{-1} is a *concentration matrix*.

Concentration models

Let $G = (V, E)$ be a simple undirected graph with $|V(G)| = m$. A *concentration model* of G is the model $\Theta = \mathbb{R}^m \times \Theta_2$ where

$$\Theta_2 = \{\Sigma \in \mathrm{PD}_m : (\Sigma)^{-1}_{ij} = 0 \text{ if } ij \notin E(G) \text{ and } i \neq j\}.$$

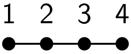
Proposition (A., Hoșten)

Let Θ_2 be a concentration model of some graph G. For a point $\Sigma \in \Theta_2$, its logaritmic Voronoi cell is equal to its log-normal spectrahedron.

In fact, we can describe $\log \mathrm{Vor}_\Theta(\Sigma)$ as:

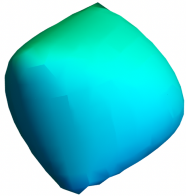
$$\log \mathrm{Vor}_\Theta(\Sigma) = \{S \in \mathrm{PD}_m : \Sigma_{ij} = S_{ij} \text{ for all } ij \in E(G) \text{ and } i = j\}.$$

Example

The concentration model of  is defined by

$$\Theta = \{\Sigma = (\sigma_{ij}) \in \mathrm{PD}_4 : (\Sigma^{-1})_{13} = (\Sigma^{-1})_{14} = (\Sigma^{-1})_{24} = 0\}.$$

Let $\Sigma = \begin{pmatrix} 6 & 1 & \frac{1}{7} & \frac{1}{28} \\ 1 & 7 & 1 & \frac{1}{4} \\ \frac{1}{7} & 1 & 8 & 2 \\ \frac{1}{28} & \frac{1}{4} & 2 & 9 \end{pmatrix}$.



Then $\log \mathrm{Vor}_\Theta(\Sigma) = \left\{ (x, y, z) : \begin{pmatrix} 6 & 1 & x & y \\ 1 & 7 & 1 & z \\ x & 1 & 8 & 2 \\ y & z & 2 & 9 \end{pmatrix} \succ 0 \right\}$.

Bivariate correlation models

A *bivariate correlation model* is a model given by the parameter space

$$\Theta_2 = \left\{ \Sigma_x := \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} : x \in (-1, 1) \right\}.$$

Given S, the derivative of $\ell(\Sigma, S)$ is $\frac{2}{(1-x^2)^2} \cdot f(x)$ where

$$f(x) = x^3 - bx^2 - x(1 - 2a) - b$$

where $a = (S_{11} + S_{22})/2$ and $b = S_{12}$.

The polynomial f has three critical points in the model iff $\Delta_f(b, a) > 0$ and $a < 1/2$.

Bivariate correlation models

Given some $\Sigma_c \in \Theta_2$, what is its logarithmic Voronoi cell?

Bivariate correlation models

Given some $\Sigma_c \in \Theta_2$, what is its logarithmic Voronoi cell?

- c must be a root of $f(x)$.
- Setting $f(c) = 0$, get $a = \frac{bc^2 - c^3 + b + c}{2c}$.
- Only $S \in \mathrm{PD}_m$ satisfying this have Σ as a critical point of $\ell_n(\Sigma, S)$.
- If either $\Delta_f(b, a) \leq 0$ or $a \geq 1/2$, then $S \in \log \mathrm{Vor}_{\Theta_2}(\Sigma)$.
- If $\Delta_f(b, a) > 0$ and $a < 1/2$, let c_1 and c_2 be the other roots of $f(x)$.
- In this case, $S \in \log \mathrm{Vor}_{\Theta}(\Sigma)$ iff $\ell_n(\Sigma_c, S) \geq \ell_n(\Sigma_{c_i}, S)$ for $i = 1, 2$.

Proposition (A., Hoșten)

Logarithmic Voronoi cells of bivariate correlation models are, in general, not equal to their log-normal spectrahedra.

Equicorrelation models

An *equicorrelation model*, denoted by E_m, is given by the parameter space

$$\Theta_2 = \{\Sigma_x \in \mathrm{Sym}(\mathbb{R}^m) : \Sigma_{ii} = 1, \Sigma_{ij} = x \text{ for } i \neq j, i, j \in [m], x \in \mathbb{R}\} \cap \mathrm{PD}_m \,.$$

How do we find the logarithmic Voronoi cell of Σ_c?

- For every S, consider the *symmetrized sample covariance matrix*

$$\bar{S} = \frac{1}{m!} \sum_{P \in S_m} PSP^T.$$

- Note $\bar{S}_{ii} = a$ and $\bar{S}_{ij} = b$ whenever $i \neq j$, and $\langle S, \Sigma_x^{-1} \rangle = \langle \bar{S}, \Sigma_x^{-1} \rangle$.
- The critical points for a general $\bar{S}$ with $\bar{S}_{ii} = a$ and $\bar{S}_{ij} = b$ fot $i \neq j$ is given by the points Σ_r where r is a root of the cubic

$$f_m(x) = (m-1)x^3 + ((m-2)(a-1) - (m-1)b)x^2 + (2a-1)x - b.$$

- Set $f_m(c) = 0$ to get the relationship between a and b that any $\bar{S} \in \log \mathrm{Vor}_{E_m}(\Sigma_c)$ must satisfy.

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- If $\Delta_{f,m}(b, a) < 0$, then $\bar{S} \in \log \mathrm{Vor}_\Theta(\Sigma_c)$.
- If $\Delta_{f,m}(b, a) \geq 0$, we might have to evaluate $\ell(\bullet, \bar{S})$, at the other two roots of f_m, and compare it to $\ell(\Sigma_c, \bar{S})$.
- These inequalities are expressions in b only.

Proposition

Logarithmic Voronoi cells of equicorrelation models are, in general, not equal to their log-normal spectrahedra.

In statistical practice, the matrices $\bar{S}$ with three critical points in the model are rare, even for small sample sizes [Amendola, Zwernik]. So we may approximate log-Voronoi cells by log-normal spectrahedra.

Thanks!